正六角形グリッドから得られるケルト結び目模様

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1. Introduction

Celtic knots are traditional geometric symbol of the Celtic peoples of ancients Britain, Scotland and Ireland. They are stylized, interlaced patterns, representing ropes or threads tied in a knot. They also appear in the art ([2]). For example, a bible manuscript called by The Book of Kells is decorated by many Celtic knots. Fisher-Mellor [1] defined *knotwork design* as a kind of alternating link diagrams related to Celtic knot. In Definition 2.2, we define *Celtic knot design* as a generalized knotwork design. Let G be a closed subset of the plane arranged by $p \times q$ regular hexagons "vertically and horizontally" without any space (see Definition 2.4). In this paper, we focus on Celtic knot design induced from G. Furthermore, let D be a link diagram representing a Celtic knot design induced from G. Let $\mathcal{K}$ be the set of the knot diagrams corresponding to the components of D, where each element of $\mathcal{K}$ is arranged so as to superimpose D on G. We say that an element of $\mathcal{K}$ is *Spur* if the element passes through the regular hexagon at upper-left corner of G (see Definition 2.5). The set excepted for Spur from $\mathcal{K}$ is called *Track* (see Definition 2.6). In Theorem 3.1, we give the number of components of a link obtained from D. In Theorem 3.3, we give that Spur passes through the regular hexagon at the lower-right corner of G. In Theorem 3.4, we give that any element of Track is uniquely determined up to ambient isotopy.

2. Preliminaries

Definition 2.1 (grid)**.** Suppose that the plane (in general surfaces) is divided into polygons. A closed subset of this plane whose boundary consisting of some edges of the polygons is called a *grid*.

Fact 2.1. *One type of regular polygon tiling the plane is a regular triangle, a square, or a regular hexagon.*

Definition 2.2 (Celtic knot projection, Celtic knot design)**.** For any grid, fix a mid-point of each edge of every regular polygon forming the grid, draw a new regular polygon inscribed at these points. The union of these new polygons can be regarded as a projection of a link. The projection is called *Celtic knot projection* (*CKP*). A link diagram obtained from the projection by adding alternating upper/lower information for each double point is called *Celtic knot design* (*CKD*).

Remark 2.1. For any grid G, there exist two types of CKD, D and D^*, induced from G. For a CKD D, D^* is the mirror image of D as in Figure 2.1 and Figure 2.2.

In this paper, we use the type of CKD as in Figure 2.1.

2000 Mathematics Subject Classification: 57R42, 05C12, 57M99.
Keywords: Celtic knot design, Celtic knot, alternating knot, plane graph, diagram, knotwork design.
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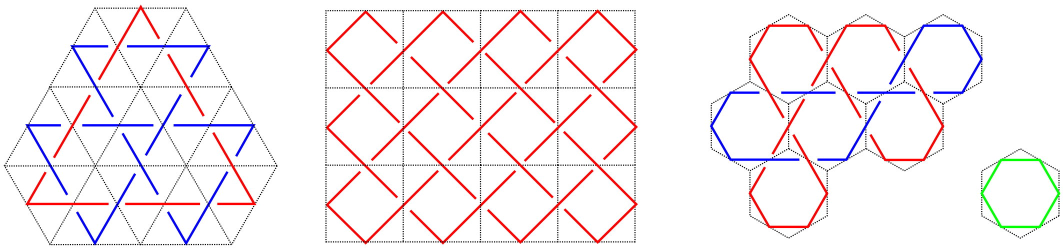

Figure 2.1: Celtic knot designs.

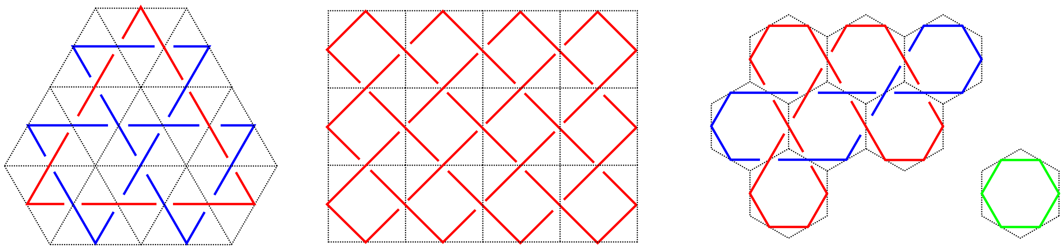

Figure 2.2: Mirror images of Celtic knot designs as in Figure 2.1.

Definition 2.3 (regular triangle grid, square grid, honeycomb grid)**.** We say that a grid is a *regular triangle grid* (a *square grid*, a *honeycomb grid* resp.) if the grid is tiled by a regular triangle (a square, a regular hexagon resp.)

Definition 2.4 ($p \times q$ honeycomb grid)**.** We consider an oblique coordinate with an angle of $\frac{\pi}{3}$ degrees on the plane. For any element of $\{(x, y) | 1 \leq y \leq p, 1 \leq x \leq q, x, y \in \mathbb{N}\}$, draw a regular hexagon whose length of one side is $\frac{1}{\sqrt{3}}$, centered at point (x, y) as in Figure 2.3. The regular hexagons are regarded as a honeycomb grid. Then, the honeycomb grid is called $p \times q$ *honeycomb grid* .

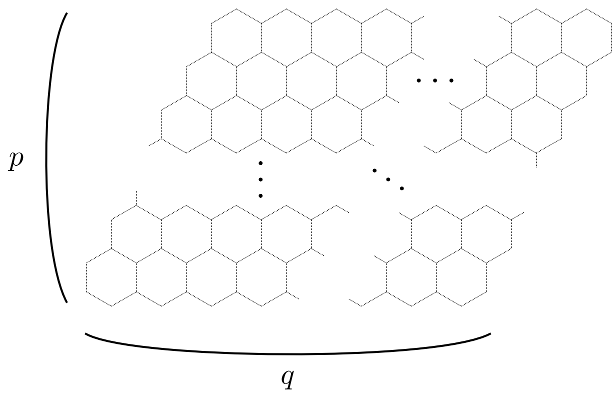

Figure 2.3: $p \times q$ honeycomb grid.

Let G be a $p \times q$ honeycomb grid and D a Celtic knot design induced from G.

Notation 2.1. Any regular hexagon of G with center coordinates (i, j) is denoted by (i, j) simply. For example, the regular hexagon at the upper-left corner of G is denoted by $(1, p)$, and the regular hexagon at the lower-right corner of G is denoted by $(q, 1)$.

Let $\mathcal{K}$ be the set of the knot diagrams corresponding to the components of D, where each element of $\mathcal{K}$ is arranged so as to superimpose D on G.

Definition 2.5 (Spur)**.** We say that an element of $\mathcal{K}$ is *Spur* if the element passes through $(1, p)$ as in Figure 2.4.

Definition 2.6 (Track)**.** The set excepted for the Spur from $\mathcal{K}$ is called *Track*.

Each element of $\mathcal{K}$ passes through at least one regular hexagon on right side of G. Let (q, j) be each hexagon on the right side of G which the Spur passes through. Then r denotes the maximum number of $\{j\}$ as in Figure 2.4.

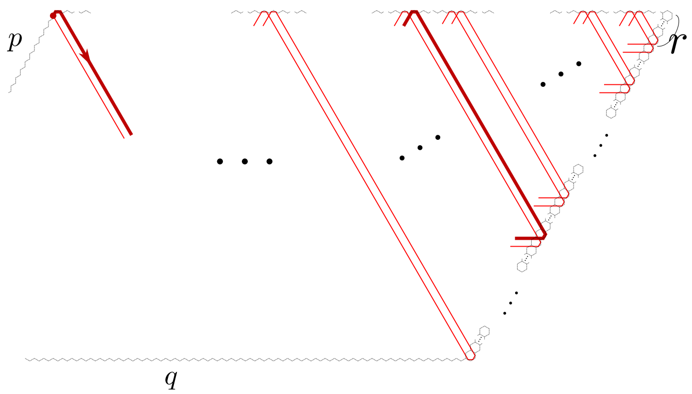

Figure 2.4: The red line represents the Spur on $p \times q$ honeycomb grid.

3. Main results

Let G be a $p \times q$ honeycomb grid and D a Celtic knot design induced from G.

Theorem 3.1. *The number of components of a link obtained from D is r.*

Proposition 3.2. *Suppose that $q = m(p + 1) - 1$ $(m = 1, 2, 3, \dots)$. Then the number of components of links obtained from D is p and each component is a trivial knot.*

Theorem 3.3. *The Spur on G passes through the regular hexagon $(q, 1)$ at the lower-right corner as in Figure 3.5.*

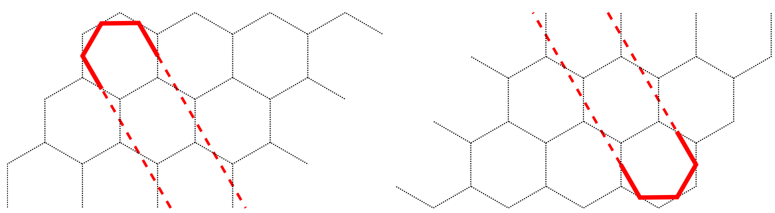

Figure 3.5: Red lines represent the Spur.

Theorem 3.4. *Any element of the Track on G is uniquely determined up to ambient isotopy.*

For example, we consider a CKD induced from 7×11 honeycomb grid as in Figure 3.6. The Spur passes through the regular hexagon $(11, 1)$ at the lower-right corner as in Figure 3.7. On the other hand, the elements of the Track are unique up to ambient isotopy as in Figure 3.8.

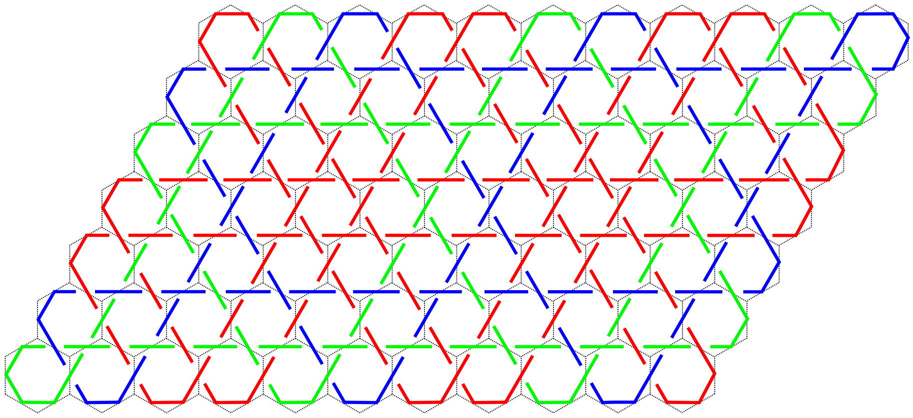

Figure 3.6: A CKD induced from 7×11 honeycomb grid.

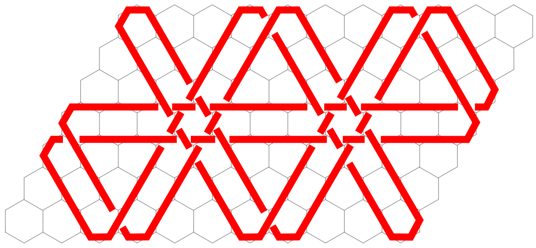

Figure 3.7: The Spur on 7×11 honeycomb grid.

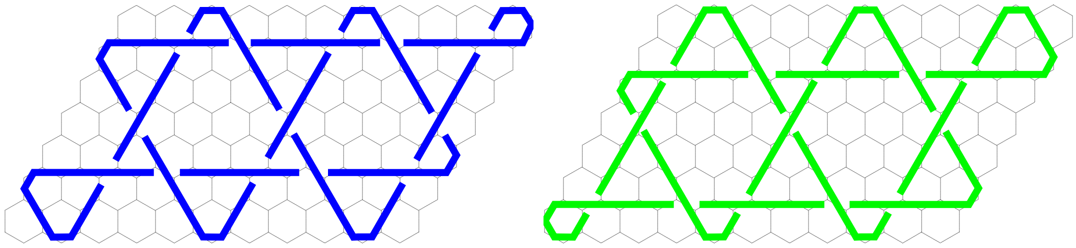

Figure 3.8: The elements of the Track on 7×11 honeycomb grid.

References

[1] Gwen Fisher and Blake Mellor, *On the Topology of Celtic Knot Designs*, Bridges Mathematical Connections in Art, Music, and Sicence, (2004), pp37–44.

[2] Aidan Meehan, *Celtic Design, Knotwork, The Secret Method of the Scribes*, Thames and Hudson, (1991).