A Concept Map for Book 1 of Euclid's *Elements*

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Abstract

Book 1 of Euclid's *Elements* begins with just a few simple assumptions and culminates in a profound statement about our universe — the Pythagorean Theorem. We have created a concept map of Book 1 designed to illustrate graphically this remarkable logical sequence. We hope that our effort, although preliminary, will be of interest to math teachers, devotees of the history of math, and anyone who deals with the graphical display of relational data.

Introduction

Although admittedly more admired than read, Euclid's *Elements of Geometry* [1] stands as the model for every branch of knowledge that has aspired to establish its tenets upon the solid foundations of deductive reasoning. Following the progression of how, from just a few simple assumptions, we can obtain a firm grasp of a deep and abstract concept is a thrilling endeavor and, indeed, the *Elements* is rightfully regarded as one of the great triumphs of the human mind. Having recently been inspired to leave the crowd of mere admirers and enter into the circle of Euclid's readers, we set ourselves the goal of copying out in Greek a single proposition each day, or whenever leisure afforded us this delightful luxury. Euclid's *Elements* comprises thirteen books that range from plane geometry to number theory; we began our project with Book 1 which carries the heading "Fundamentals of Plane Geometry Involving Straight-Lines" and which culminates in Euclid's famous proof of the Pythagorean Theorem. Even more so than the actual content of its 48 propositions, Euclid's Book 1 is remarkable for the tight logical structure that interconnects these propositions within a coherent whole. By carefully noting the dependencies of each successive proposition, it occurred to us that some form of infographic, or concept map, might be able to convey the logical structure of Book 1 in a novel and compelling way. We hope that the concept map we have produced (Figure 1), although preliminary, will be of interest to math teachers, devotees of the history of math, and anyone who deals with the graphical display of relational data.

The Structure of Book 1

Euclid's Book 1 comprises 48 numbered propositions preceded by a list of 23 definitions, 5 postulates and 5 axioms. Euclid's *Elements* is generally considered to be the original exemplar of an axiomatic system but it does not, in fact, make use of the Greek word ἀξίωμα ("axiom"). Instead, Euclid uses the phrase κοιναὶ ἔννοιαι, ("common notions") to refer to the five assumptions that subsequent tradition has called "axioms." This is important because insofar as both the postulates and the axioms serve to establish the "rules of the game", both could be called axioms in a modern context. However, Euclid's separation of the postulates from the axioms is a crucial one: Whereas the axioms indicate which logical operations are assumed (for example, that "Things equal to the same thing are also equal to one another" — Axiom 1), the postulates indicate which *technological* operations

are assumed (for example, "To draw a straight-line from any point to any point" — Postulate 1). In this context it becomes clear why Euclid's famous fifth postulate — that non-parallel lines, when extended, will intersect each other — is rightly classed as a postulate and not an axiom as it has been in some editions: This postulate is a technological tool that must actually be applied in the construction of certain diagrams (namely, in the proofs of propositions 1.29 and 1.44).

The distinction between the technological postulates and the logical axioms continues within the 48 propositions themselves. Fourteen of the 48 propositions are technological constructions (for example, "To construct an equilateral triangle on a given finite straight-line" — Proposition 1) whereas the remaining 34 propositions are logical inferences deduced from the diagrams we can construct (for example, "For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another" — Proposition 5).

The blanket term "Proposition" does not come from Euclid since in Greek the propositions simply appear as a numbered list without any headings. Nevertheless, Euclid makes the distinction between these two classes of propositions perfectly clear in how their proofs are concluded. The 14 technological constructions each conclude with the phrase ὅπερ ἔδει ποιῆσαι ("what was necessary to do") while the 34 logical inferences conclude with the phrase ὅπερ ἔδει δεῖξαι ("what was necessary to show"). Unfortunately, in its transmission through Latin and then into modern languages, these phrases are usually contracted to the catch-all Q.E.D (*quod erat demonstrandum*, "that which was to be shown") thereby erasing this important clue about the two types of propositions. Although some English editions of Euclid preserve the distinction by calling one type "problems" and the other type "theorems", we prefer the terms "technological constructions" and "logical inferences" — or just "technologies" and "inferences" for short — as these terms emphasize the distinction more clearly.

While the 48 propositions of Book 1 each demonstrate something of intrinsic interest (be it a technological construction or a logical inference), Book 1 shouldn't be thought of as hodgepodge of random geometrical proofs. Rather, the structure of the book is a logical progression with nearly every proposition used to prove a subsequent one. The book's grand finale is Euclid's proof of the Pythagorean Theorem (Proposition 47) and its converse (Proposition 48). Inasmuch as all the propositions are so tightly interconnected, Book 1 of Euclid's *Elements* reads almost like a mathematical poem. Devising a means to showcase the beauty of Book 1 to a broader audience is what inspired us to attempt to map its structure graphically.

Creating a Concept Map

Employing the power of graphics to facilitate a grasp Euclid's *Elements* has at least one famous precedent. In 1847, Olivier Byrne produced an English language edition of Euclid where all the lines and figures were printed in beautiful colors such that they did not need to be identified using letters; instead, small copies of the shapes themselves could be inserted directly into the proofs of each proposition [2]. But whereas Byrne wanted to devise a system that would make it easier to understand the individual propositions, we wish to illustrate the logical structure of Book 1 considered as a whole. All the same, in attempting to visualize something so information-dense, we tried to remain mindful of the advice that, "What is to be sought in designs for the display of information is the clear portrayal of complexity. Not the complication of the simple." [3]

In creating a concept map for Book 1 of Euclid, our overarching strategy was to produce an illustration that emphasized (1) the distinction between the technological and inferential propositions and (2) the logical progression of how the propositions build upon each other.

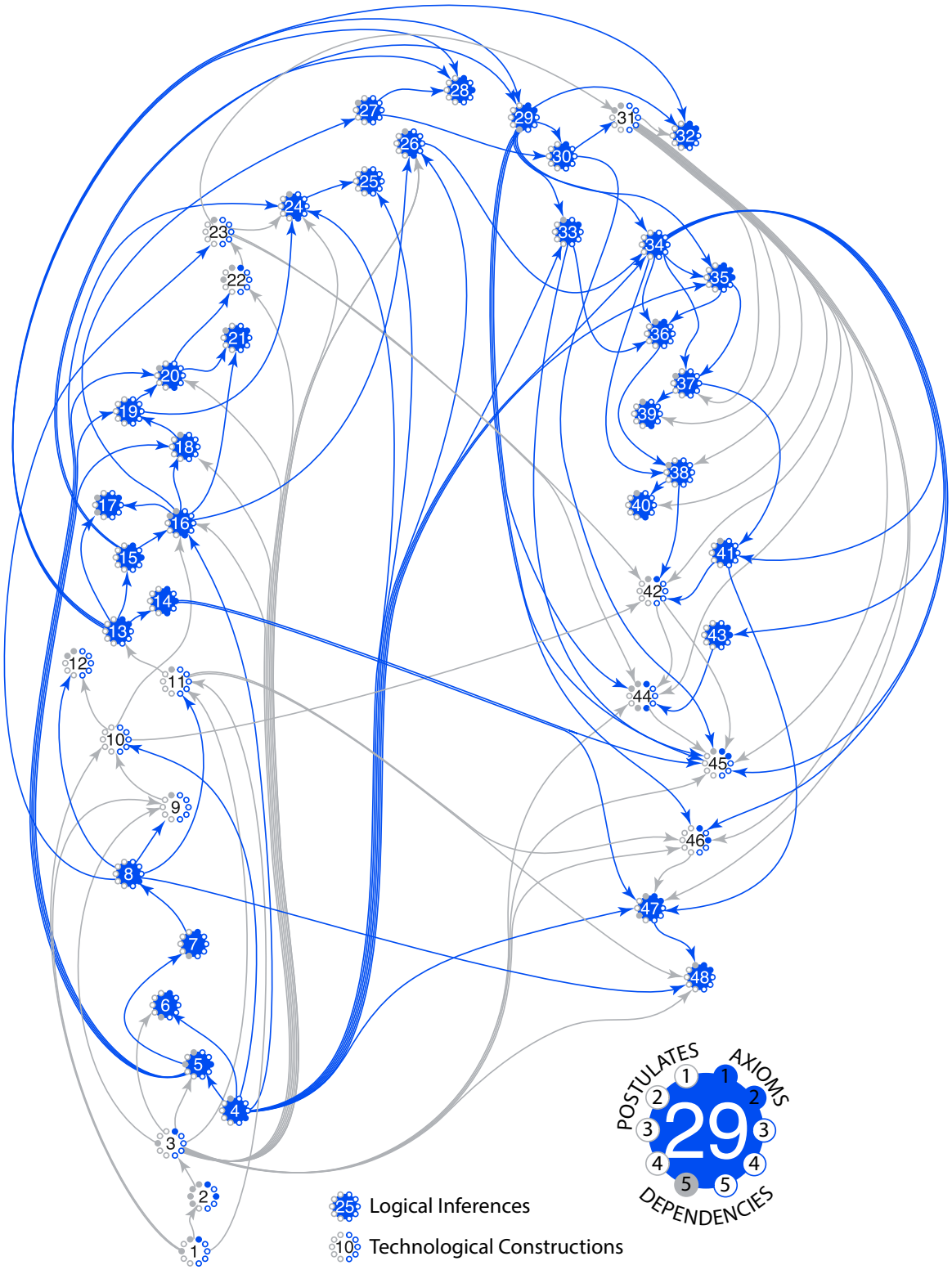

Figure 1 : *Concept Map for Book 1 of Euclid's* Elements.

The resulting concept map (Figure 1) was produced from a simple spreadsheet that marked the dependencies of each of the propositions on the postulates, axioms and preceding propositions. This spreadsheet was then imported into the R programming language [4] where the `Rgraphviz` package [5] was used to create a directed graph. This vector output was then imported into Adobe Illustrator where the diagram was modified by hand in order to produce a map that had, at least for us, a subjective visual appeal.

Each of the 48 propositions is depicted as a numbered circle; the 34 logical inferences have a dark shading and the 14 technological constructions have no shading. Dependencies between the propositions are rendered by appropriately shaded (dark and light) arrows. Surrounding the propositions are 10 small circle indicators representing the 5 postulates (white) and 5 axioms (dark); a dependency is indicated when the circle is filled. The shading scheme of the indicators intentionally mirrors the logical and technological categories that, we assert, also characterize the propositions.

Conclusion

Figure 1 is our preliminary attempt to map the logical structure of Book 1 of Euclid's *Elements*. This concept map is designed to emphasize how the propositions fall into two distinct classes that mirror the postulates and axioms. The map is also designed to convey the logical motion (clockwise) of the book by illustrating how subsequent propositions are built from earlier ones. Nevertheless, many features that we wished to include, such as grouping the propositions according to content (e.g., propositions about triangles, propositions about parallel lines, etc.), were not feasible on this attempt without transgressing the already strained limits of clarity. All the same, it is our hope that this concept map offers up a unique way of visualizing an important mathematical text, and can serve to spark discussion — or, better yet, copycats — amongst teachers, students and Euclid aficionados wherever they may be. Which was the very thing we set out to do: ὅπερ ἔδει ποιῆσαι!

References

[1] Euclid, *Euclid's Elements of Geometry*, the Greek text of J. L. Heiberg edited and provided with a modern English translation by Richard Fitzpatrick, revised edition (2008), Lulu Press. `http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf` (as of Mar. 15, 2015).

[2] Oliver Byrne, *The First Six Books of The Elements of Euclid In Which Colored Diagrams and Symbols are Used Instead of Letters for the Greater Ease of Learners*, originally published by William Pickering (1847), reprinted facsimile edition (2013), Taschen. `https://archive.org/stream/firstsixbooksofe00byrn#page/n5/mode/2up` (as of Mar. 15, 2015).

[3] Edward R. Tufte, *The Visual Display of Quantitative Information*, second edition (2001), Graphics Press, p. 191.

[4] R Core Team (2013). *R: A language and environment for statistical computing.* R Foundation for Statistical Computing, Vienna, Austria. `http://www.R-project.org/`

[5] Kasper Daniel Hansen, Jeff Gentry, Li Long, Robert Gentleman, Seth Falcon, Florian Hahne and Deepayan Sarkar. *Rgraphviz: Provides plotting capabilities for R graph objects.* R package version 2.10.0.

Here, EEI stands for Book I of Euclidâ€™s â€œElementsâ€. 1. Mowaï¬€aq Hajja and Horst Martini. Propositions referred to in the text by the code EEI.xxx, EEI. (18+19), 20, 21(a), and 21(b) are from Euclidâ€™s Elements. Prop. Statement Figure.Â angles are less than 120â—¦each. 12. Proposition 21 of Book I of Euclidâ€™s Elements. 5 Characterization of the interior points of a triangle and an open. problem.