An Introduction to the Wavelet Analysis of Time Series

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Overview

- wavelets are analysis tools mainly for
  - time series analysis (focus of this tutorial)
  - image analysis (will not cover)
- as a subject, wavelets are
  - relatively new (1983 to present)
  - synthesis of many new/old ideas
  - keyword in 10,558+ articles & books since 1989
    (2000+ in the last year alone)
- broadly speaking, have been two waves of wavelets
  - continuous wavelet transform (1983 and on)
  - discrete wavelet transform (1988 and on)
Game Plan

• introduce subject via CWT

• describe DWT and its main ‘products’
  – multiresolution analysis (additive decomposition)
  – analysis of variance (‘power’ decomposition)

• describe selected uses for DWT
  – wavelet variance (related to Allan variance)
  – decorrelation of fractionally differenced processes (closely related to power law processes)
  – signal extraction (denoising)
What is a Wavelet?

- wavelet is a ‘small wave’ (sinusoids are ‘big waves’)
- real-valued $\psi(t)$ is a wavelet if
  1. integral of $\psi(t)$ is zero: $\int_{-\infty}^{\infty} \psi(t) \, dt = 0$
  2. integral of $\psi^2(t)$ is unity: $\int_{-\infty}^{\infty} \psi^2(t) \, dt = 1$
     (called ‘unit energy’ property)
- wavelets so defined deserve their name because
  - #2 says we have, for every small $\epsilon > 0$,
    $$\int_{-T}^{T} \psi^2(t) \, dt < 1 - \epsilon,$$
    for some finite $T$ (might be quite large!)
  - length of $[-T,T]$ small compare to $[-\infty, \infty]$
  - #2 says $\psi(t)$ must be nonzero somewhere
  - #1 says $\psi(t)$ balances itself above/below 0
- Fig. 1: three wavelets
- Fig. 2: examples of complex-valued wavelets
Basic of Wavelet Analysis: I

- wavelets tell us about variations in local averages
- to quantify this description, let \( x(t) \) be a ‘signal’
  - real-valued function of \( t \)
  - will refer to \( t \) as time (but can be, e.g., depth)
- consider average value of \( x(t) \) over \([a, b]\):
  \[
  \frac{1}{b-a} \int_a^b x(u) \, du \equiv \alpha(a, b)
  \]
- reparameterize in terms of \( \lambda \) & \( t \)
  \[
  A(\lambda, t) \equiv \alpha(t - \frac{\lambda}{2}, t + \frac{\lambda}{2}) = \frac{1}{\lambda} \int_{t-\frac{\lambda}{2}}^{t+\frac{\lambda}{2}} x(u) \, du
  \]
  - \( \lambda \equiv b - a \) is called scale
  - \( t = (a + b)/2 \) is center time of interval
- \( A(\lambda, t) \) is average value of \( x(t) \) over scale \( \lambda \) at \( t \)
Basics of Wavelet Analysis: II

- average values of signals are of wide-spread interest
  - hourly rainfall rates
  - monthly mean sea surface temperatures
  - yearly average temperatures over central England
  - etc., etc., etc. (Rogers & Hammerstein, 1951)

- Fig. 3: fractional frequency deviates in clock 571
  - can regard as averages of form \([t - \frac{1}{2}, t + \frac{1}{2}]\)
  - \(t\) is measured in days (one measurement per day)
  - plot shows \(A(1, t)\) versus integer \(t\)
  - \(A(1, t) = 0 \Rightarrow\) master clock & 571 agree perfectly
  - \(A(1, t) < 0 \Rightarrow\) clock 571 is losing time
  - can easily correct if \(A(1, t)\) constant
  - quality of clock related to changes in \(A(1, t)\)
Basics of Wavelet Analysis: III

• can quantify changes in $A(1, t)$ via

$$D(1, t - \frac{1}{2}) \equiv A(1, t) - A(1, t - 1) = \int_{t - \frac{1}{2}}^{t + \frac{1}{2}} x(u) \, du - \int_{t - \frac{3}{2}}^{t - \frac{1}{2}} x(u) \, du,$$

or, equivalently,

$$D(1, t) = A(1, t + \frac{1}{2}) - A(1, t - \frac{1}{2}) = \int_{t}^{t+1} x(u) \, du - \int_{t-1}^{t} x(u) \, du$$

• generalizing to scales other than unity yields

$$D(\lambda, t) \equiv A(\lambda, t + \frac{\lambda}{2}) - A(\lambda, t - \frac{\lambda}{2}) = \frac{1}{\lambda} \int_{t}^{t+\lambda} x(u) \, du - \frac{1}{\lambda} \int_{t-\lambda}^{t} x(u) \, du$$

• $D(\lambda, t)$ often of more interest than $A(\lambda, t)$

• can connect to Haar wavelet: write

$$D(\lambda, t) = \int_{-\infty}^{\infty} \tilde{\psi}_{\lambda,t}(u) x(u) \, du$$

with

$$\tilde{\psi}_{\lambda,t}(u) \equiv \begin{cases} -1/\lambda, & t - \lambda \leq u < t; \\ 1/\lambda, & t \leq u < t + \lambda; \\ 0, & \text{otherwise}. \end{cases}$$
Basics of Wavelet Analysis: IV

- specialize to case $\lambda = 1$ and $t = 0$:
  \[ \tilde{\psi}_{1,0}(u) \equiv \begin{cases} 
  -1, & -1 \leq u < 0; \\
  1, & 0 \leq u < 1; \\
  0, & \text{otherwise.} 
  \end{cases} \]
  comparison to $\psi^H(u)$ yields $\tilde{\psi}_{1,0}(u) = \sqrt{2}\psi^H(u)$

- Haar wavelet mines out info on difference between unit scale averages at $t = 0$ via
  \[ \int_{-\infty}^{\infty} \psi^H(u) x(u) \, du \equiv W^H(1, 0) \]

- to mine out info at other $t$’s, just shift $\psi^H(u)$:
  \[ \psi^H_{1,t}(u) \equiv \psi^H(u-t); \text{ i.e., } \psi^H_{1,t}(u) = \begin{cases} 
  -\frac{1}{\sqrt{2}}, & t - 1 \leq u < t; \\
  \frac{1}{\sqrt{2}}, & t \leq u < t + 1; \\
  0, & \text{otherwise} 
  \end{cases} \]
  Fig. 4: top row of plots

- to mine out info about other $\lambda$’s, form
  \[ \psi^H_{1,t}(u) \equiv \frac{1}{\sqrt{\lambda}} \psi^H \left( \frac{u-t}{\lambda} \right) = \begin{cases} 
  -\frac{1}{\sqrt{2\lambda}}, & t - \lambda \leq u < t; \\
  \frac{1}{\sqrt{2\lambda}}, & t \leq u < t + \lambda; \\
  0, & \text{otherwise.} 
  \end{cases} \]
  Fig. 4: bottom row of plots
Basics of Wavelet Analysis: V

- can check that $\psi^H_{\lambda,t}(u)$ is a wavelet for all $\lambda & t$
- use $\psi^H_{\lambda,t}(u)$ to obtain
  \[ W^H(\lambda, t) \equiv \int_{-\infty}^{\infty} \psi^H_{\lambda,t}(u)x(u) \, du \propto D(\lambda, t) \]
  left-hand side is Haar CWT
- can do the same with other wavelets:
  \[ W(\lambda, t) \equiv \int_{-\infty}^{\infty} \psi_{\lambda,t}(u)x(u) \, du, \text{ where } \psi_{\lambda,t}(u) \equiv \frac{1}{\sqrt{\lambda}}\psi \left( \frac{u - t}{\lambda} \right) \]
  left-hand side is CWT based on $\psi(u)$
- interpretation for $\psi^{fdG}(u)$ and $\psi^{Mh}(u)$ (Fig. 1):
  differences of adjacent weighted averages
Basics of Wavelet Analysis: VI

- basic CWT result: if $\psi(u)$ satisfies admissibility condition, can recover $x(t)$ from its CWT:

$$x(t) = \frac{1}{C_\psi} \int_{0}^{\infty} \left[ \int_{-\infty}^{\infty} W(\lambda, t) \frac{1}{\sqrt{\lambda}} \psi \left( \frac{t - u}{\lambda} \right) du \right] \frac{d\lambda}{\lambda^2},$$

where $C_\psi$ is constant depending just on $\psi$

- conclusion: $W(\lambda, t)$ equivalent to $x(t)$

- can also show that

$$\int_{-\infty}^{\infty} x^2(t) \, dt = \frac{1}{C_\psi} \left[ \int_{0}^{\infty} \int_{-\infty}^{\infty} W^2(\lambda, t) \, dt \right] \frac{d\lambda}{\lambda^2}$$

- LHS called energy in $x(t)$
- RHS integrand is energy density over $\lambda$ & $t$

- Fig. 3: Mexican hat CWT of clock 571 data
Beyond the CWT: the DWT

- critique: have transformed signal into an image
- can often get by with subsamples of $W(\lambda, t)$
- leads to notion of discrete wavelet transform (DWT)
  - can regard as dyadic ‘slices’ through CWT
  - can further subsample slices at various $t$’s
- DWT has appeal in its own right
  - most time series are sampled as discrete values
    (can be tricky to implement CWT)
  - can formulate as orthonormal transform
    (facilitates statistical analysis)
  - approximately decorrelates certain time series
    (including power law processes)
  - standardization to dyadic scales often adequate
  - can be faster than the fast Fourier transform!
- will concentrate on DWT for remainder of tutorial
Overview of DWT

• let $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]^T$ be observed time series (for convenience, assume $N$ integer multiple of $2^{J_0}$)

• let $\mathcal{W}$ be $N \times N$ orthonormal DWT matrix

• $\mathbf{W} = \mathcal{W}\mathbf{X}$ is vector of DWT coefficients

• orthonormality says $\mathbf{X} = \mathcal{W}^T\mathbf{W}$, so $\mathbf{X} \Leftrightarrow \mathbf{W}$

• can partition $\mathbf{W}$ as follows:

$$\mathbf{W} = \begin{bmatrix}
\mathbf{W}_1 \\
\vdots \\
\mathbf{W}_{J_0} \\
\mathbf{V}_{J_0}
\end{bmatrix}$$

• $\mathbf{W}_j$ contains $N_j = N/2^j$ wavelet coefficients
  – related to changes of averages at scale $\tau_j = 2^{j-1}$ ($\tau_j$ is $j$th ‘dyadic’ scale)
  – related to times spaced $2^j$ units apart

• $\mathbf{V}_{J_0}$ contains $N_{J_0} = N/2^{J_0}$ scaling coefficients
  – related to averages at scale $\lambda_{J_0} = 2^{J_0}$
  – related to times spaced $2^{J_0}$ units apart
Example: Haar DWT

• Fig. 5: $W$ for Haar DWT with $N = 16$
  – first 8 rows yield $W_1 \propto changes$ on scale 1
  – next 4 rows yield $W_2 \propto changes$ on scale 2
  – next 2 rows yield $W_3 \propto changes$ on scale 4
  – next to last row yields $W_4 \propto change$ on scale 8
  – last row yields $V_4 \propto average$ on scale 16

• Fig. 6: Haar DWT coefficients for clock 571
DWT in Terms of Filters

• filter $X_0, X_1, \ldots, X_{N-1}$ to obtain

\[ 2^{j/2} \tilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} h_{j,l} X_{t-l \mod N}, \quad t = 0, 1, \ldots, N - 1 \]

where $h_{j,l}$ is $j$th level wavelet filter

– note: circular filtering

• subsample to obtain wavelet coefficients:

\[ W_{j,t} = 2^{j/2} \tilde{W}_{j,2^j(t+1)-1}, \quad t = 0, 1, \ldots, N_j - 1, \]

where $W_{j,t}$ is $t$th element of $W_j$

• Figs. 7 & 8: Haar, D(4), C(6) & LA(8) wavelet filters

• $j$th wavelet filter is band-pass with pass-band $[\frac{1}{2^j+1}, \frac{1}{2^j}]$

• note: $j$th scale related to interval of frequencies

• similarly, scaling filters yield $V_{J_0}$

• Figs. 9 & 10: Haar, D(4), C(6) & LA(8) scaling filters

• $J_0$th scaling filter is low-pass with pass-band $[0, \frac{1}{2^{J_0+1}}]$
Pyramid Algorithm: I

• can formulate DWT via ‘pyramid algorithm’
  – elegant iterative algorithm for computing DWT
  – implicitly *defines* $\mathcal{W}$
  – computes $\mathbf{W} = \mathbf{WX}$ using $O(N)$ multiplications
    * ‘brute force’ method uses $O(N^2)$
    * FFT algorithm uses $O(N \log_2 N)$

• algorithm makes use of two basic filters
  – wavelet filter $h_l$ of unit scale $h_l \equiv h_{1,l}$
  – associated scaling filter $g_l$
The Wavelet Filter: I

• let $h_l, l = 0, \ldots, L - 1$, be a real-valued filter
  - $L$ is filter width so $h_0 \neq 0 \& h_{L-1} \neq 0$
  - $L$ must be even
  - assume $h_l = 0$ for $l < 0 \& l \geq L$

• $h_l$ called a wavelet filter if it has these 3 properties
  1. summation to zero:
     \[
     \sum_{l=0}^{L-1} h_l = 0
     \]
  2. unit energy:
     \[
     \sum_{l=0}^{L-1} h_l^2 = 1
     \]
  3. orthogonality to even shifts:
     \[
     \sum_{l=0}^{L-1} h_l h_{l+2n} = \sum_{l=-\infty}^{\infty} h_l h_{l+2n} = 0
     \]
     for all nonzero integers $n$

• 2 & 3 together called orthonormality property
The Wavelet Filter: II

• transfer & squared gain functions for $h_l$:

$$H(f) \equiv \sum_{l=0}^{L-1} h_le^{-2\pi fl} \quad \& \quad \mathcal{H}(f) \equiv |H(f)|^2$$

• can argue that orthonormality property equivalent to

$$\mathcal{H}(f) + \mathcal{H}(f + \frac{1}{2}) = 2 \quad \text{for all } f$$

• Fig. 11: $\mathcal{H}(f)$ for Daubechies wavelet filters
  
  – $L = 2$ case is Haar wavelet filter
  
  – filter cascade with averaging & differencing filters
  
  – high-pass filter with pass-band $[\frac{1}{4}, \frac{1}{2}]$
  
  – can regard as half-band filter
The Scaling Filter: I

- scaling filter: \( g_l \equiv (-1)^{l+1} h_{L-1-l} \)
  - reverse \( h_l \) & flip sign of every other coefficient
  - e.g.: \( h_0 = \frac{1}{\sqrt{2}} \) & \( h_1 = -\frac{1}{\sqrt{2}} \) \( \Rightarrow g_0 = g_1 = \frac{1}{\sqrt{2}} \)
  - \( g_l \) is ‘quadrature mirror’ filter for \( h_l \)

- properties of \( h_l \) imply \( g_l \) has these properties:
  1. summation to \( \pm \sqrt{2} \), so will assume
     \[ \sum_{l=0}^{L-1} g_l = \sqrt{2} \]
  2. unit energy:
     \[ \sum_{l=0}^{L-1} g_l^2 = 1 \]
  3. orthogonality to even shifts:
     \[ \sum_{l=0}^{L-1} g_l g_{l+2n} = \sum_{l=-\infty}^{\infty} g_l g_{l+2n} = 0 \]
     for all nonzero integers \( n \)
  4. orthogonality to wavelet filter at even shifts:
     \[ \sum_{l=0}^{L-1} g_l h_{l+2n} = \sum_{l=-\infty}^{\infty} g_l h_{l+2n} = 0 \]
     for all integers \( n \)
The Scaling Filter: II

• transfer & squared gain functions for \( g_l \):

\[
G(f) \equiv \sum_{l=0}^{L-1} g_l e^{-i2\pi fl} \quad \text{&} \quad \mathcal{G}(f) \equiv |G(f)|^2
\]

• can argue that \( \mathcal{G}(f) = \mathcal{H}(f - \frac{1}{2}) \)
  
  – have \( \mathcal{G}(0) = \mathcal{H}(-\frac{1}{2}) = \mathcal{H}(\frac{1}{2}) \) & \( \mathcal{G}(\frac{1}{2}) = \mathcal{H}(0) \)
  
  – since \( h_l \) is high-pass, \( g_l \) must be low-pass
  
  – low-pass filter with pass-band \([0, \frac{1}{4}]\)
  
  – can also regard as half-band filter

• orthonormality property equivalent to

\[
\mathcal{G}(f) + \mathcal{G}(f + \frac{1}{2}) = 2 \quad \text{or} \quad \mathcal{H}(f) + \mathcal{G}(f) = 2 \quad \text{for all} \ f
\]
Pyramid Algorithm: II

• define $V_0 \equiv X$ and set $j = 1$

• input to $j$th stage of pyramid algorithm is $V_{j-1}$
  
  – $V_{j-1}$ is full-band
  
  – related to frequencies $[0, \frac{1}{2j}]$ in $X$

• filter with half-band filters and downsample:

\[
W_{j,t} \equiv \sum_{l=0}^{L-1} h_l V_{j-1,2t+1-l \ mod \ N_{j-1}}
\]

\[
V_{j,t} \equiv \sum_{l=0}^{L-1} g_l V_{j-1,2t+1-l \ mod \ N_{j-1}};
\]

$t = 0, \ldots, N_j - 1$

• place these in vectors $W_j$ & $V_j$
  
  – $W_j$ are wavelet coefficients for scale $\tau_j = 2^{j-1}$
  
  – $V_j$ are scaling coefficients for scale $\lambda_j = 2^j$

• increment $j$ and repeat above until $j = J_0$

• yields DWT coefficients $W_1, \ldots, W_{J_0}, V_{J_0}$
Pyramid Algorithm: III

- can formulate inverse pyramid algorithm (recovers $V_{j-1}$ from $W_j$ and $V_j$)
- algorithm implicitly defines transform matrix $W$
- partition $W$ commensurate with $W_j$:

$$W = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_{J_0} \\ V_{J_0} \end{bmatrix} \quad \text{parallels} \quad W = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_{J_0} \\ V_{J_0} \end{bmatrix}$$

- rows of $W_j$ use $j$th level filter $h_{j,l}$ with DFT

$$H(2^{j-1}f) \prod_{l=0}^{j-2} G(2^l f)$$

($h_{j,l}$ has $L_j = (2^j - 1)(L - 1) + 1$ nonzero elements)

- $W_j$ is $N_j \times N$ matrix such that

$$W_j = W_jX \quad \text{and} \quad W_jW_j^T = I_{N_j}$$
Two Consequences of Orthonormality

- multiresolution analysis (MRA)
  \[ X = W^T W = \sum_{j=1}^{J_0} W_j^T W_j + V_{J_0}^T V_{J_0} \equiv \sum_{j=1}^{J_0} D_j + S_{J_0} \]
  - scale-based additive decomposition
  - \( D_j \)'s & \( S_{J_0} \) called details & smooth

- analysis of variance
  - consider ‘energy’ in time series:
    \[ \|X\|^2 = X^T X = \sum_{t=0}^{N-1} X_t^2 \]
  - energy preserved in DWT coefficients:
    \[ \|W\|^2 = \|W^T X\|^2 = X^T W^T W X = X^T X = \|X\|^2 \]
  - since \( W_1, \ldots, W_{J_0}, V_{J_0} \) partitions \( W \), have
    \[ \|W\|^2 = \sum_{j=1}^{J_0} \|W_j\|^2 + \|V_{J_0}\|^2, \]
  leading to analysis of sample variance:
    \[ \hat{\sigma}^2 \equiv \frac{1}{N} \sum_{t=0}^{N-1} (X_t - \bar{X})^2 = \frac{1}{N} \sum_{j=1}^{J_0} \|W_j\|^2 + \left( \frac{1}{N} \|V_{J_0}\|^2 - \bar{X}^2 \right) \]
  - scale-based decomposition (cf. frequency-based)
Variation: Maximal Overlap DWT

• can eliminate downsampling and use

\[ \tilde{W}_{j,t} \equiv \frac{1}{2^{j/2}} \sum_{l=0}^{L_j-1} h_{j,l} X_{t - l \mod N}, \quad t = 0, 1, \ldots, N - 1 \]

...to define MODWT coefficients \( \tilde{W}_j \) (& also \( \tilde{V}_j \))

• unlike DWT, MODWT is not orthonormal
  (in fact MODWT is highly redundant)

• like DWT, can do MRA & analysis of variance:

\[ \|X\|^2 = \sum_{j=1}^{J_0} \|\tilde{W}_j\|^2 + \|\tilde{V}_{J_0}\|^2 \]

• unlike DWT, MODWT works for all samples sizes \( N \)
  (i.e., power of 2 assumption is not required)
  – if \( N \) is power of 2, can compute MODWT
    using \( O(N \log_2 N) \) operations
    (i.e., same as FFT algorithm)
  – contrast to DWT, which uses \( O(N) \) operations

• Fig. 12: Haar MODWT coefficients for clock 571
  (cf. Fig. 6 with DWT coefficients)
Definition of Wavelet Variance

• let $X_t, t = \ldots, -1, 0, 1, \ldots$, be a stochastic process

• run $X_t$ through $j$th level wavelet filter:

$$W_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l}, \quad t = \ldots, -1, 0, 1, \ldots,$$

which should be contrasted with

$$\tilde{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} X_{t-l \mod N}, \quad t = 0, 1, \ldots, N - 1$$

• definition of time dependent wavelet variance (also called wavelet spectrum):

$$\nu^2_{X,t}(\tau_j) \equiv \text{var}\{W_{j,t}\},$$

assuming $\text{var}\{W_{j,t}\}$ exists and is finite

• $\nu^2_{X,t}(\tau_j)$ depends on $\tau_j$ and $t$

• will consider time independent wavelet variance:

$$\nu^2_X(\tau_j) \equiv \text{var}\{W_{j,t}\}$$

(can be easily adapted to time varying situation)
Rationale for Wavelet Variance

• decomposes variance on scale by scale basis
• useful substitute/complement for spectrum
• useful substitute for process/sample variance
Variance Decomposition

• suppose $X_t$ has power spectrum $S_X(f)$:

$$\int_{-1/2}^{1/2} S_X(f) \, df = \text{var}\{X_t\};$$

i.e., decomposes \(\text{var}\{X_t\}\) across frequencies $f$

  - involves uncountably infinite number of $f$’s
  - $S_X(f) \Delta f \approx$ contribution to \(\text{var}\{X_t\}\) due to $f$’s
    in interval of length $\Delta f$ centered at $f$

• wavelet variance analog to fundamental result:

$$\sum_{j=1}^{\infty} \nu^2_X(\tau_j) = \text{var}\{X_t\}$$

i.e., decomposes \(\text{var}\{X_t\}\) across scales $\tau_j$

  - recall DWT/MODWT and sample variance
  - involves countably infinite number of $\tau_j$’s
  - $\nu^2_X(\tau_j)$ contribution to \(\text{var}\{X_t\}\) due to scale $\tau_j$
  - $\nu_X(\tau_j)$ has same units as $X_t$ (easier to interpret)
Spectrum Substitute/Complement

- because $\tilde{h}_{j,l} \approx$ bandpass over $[1/2^{j+1}, 1/2^j]$

$$\nu^2_X(\tau_j) \approx 2 \int_{1/2^{j+1}}^{1/2^j} S_X(f) \, df$$

- if $S_X(f)$ ‘featureless’, info in $\nu^2_X(\tau_j) \iff$ info in $S_X(f)$

- $\nu^2_X(\tau_j)$ more succinct: only 1 value per octave band

- example: $S_X(f) \propto |f|^\alpha$, i.e., power law process
  - can deduce $\alpha$ from slope of log $S_X(f)$ vs. log $f$
  - implies $\nu^2_X(\tau_j) \propto \tau_j^{-\alpha-1}$ approximately
  - can deduce $\alpha$ from slope of log $\nu^2_X(\tau_j)$ vs. log $\tau_j$
  - no loss of ‘info’ using $\nu^2_X(\tau_j)$ rather than $S_X(f)$

- with Haar wavelet, obtain pilot spectrum estimate proposed in Blackman & Tukey (1958)
Substitute for Variance: I

• can be difficult to estimate process variance
• $\nu_2^{\mathcal{H}(\tau_j)}$ useful substitute: easy to estimate & finite
• let $\mu = E\{X_t\}$ be known, $\sigma^2 = \text{var} \{X_t\}$ unknown
• can estimate $\sigma^2$ using

$$\tilde{\sigma}^2 = \frac{1}{N} \sum_{t=0}^{N-1} (X_t - \mu)^2$$

• estimator above is unbiased: $E\{\tilde{\sigma}^2\} = \sigma^2$
• if $\mu$ is unknown, can estimate $\sigma^2$ using

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{t=0}^{N-1} (X_t - \bar{X})^2$$

• there is some (non-pathological!) $X_t$ such that

$$\frac{E\{\hat{\sigma}^2\}}{\sigma^2} < \epsilon$$

for any given $\epsilon > 0$ & $N \geq 1$
• $\hat{\sigma}^2$ can badly underestimate $\sigma^2$!
• example: power law process with $-1 < \alpha < 0$
Substitute for Variance: II

• Q: why is wavelet variance useful when $\sigma^2$ is not?
• replaces ‘global’ variability with variability over scales
• if $X_t$ stationary with mean $\mu$, then
  $$E\{W_{j,t}\} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} E\{X_{t-l}\} = \mu \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} = 0$$
  because $\sum_l \tilde{h}_{j,l} = 0$
• $E\{W_{j,t}\}$ known, so can get unbiased estimator of
  $\text{var}\{W_{j,t}\} = \nu^2_{X}(\tau_j)$
• certain nonstationary $X_t$ have well-defined $\nu^2_{X}(\tau_j)$
• example: power law processes with $\alpha \leq -1$
  (example of process with stationary increments)
Estimation of Wavelet Variance: I

- can base estimator on MODWT of \( X_0, X_1, \ldots, X_{N-1} \):
  \[
  \hat{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l}X_{t-l \mod N}, \quad t = 0, 1, \ldots, N - 1
  \]
  (DWT-based estimator possible, but less efficient)
- recall that
  \[
  \bar{W}_{j,t} \equiv \sum_{l=0}^{L_j-1} \tilde{h}_{j,l}X_{t-l}, \quad t = 0, \pm 1, \pm 2, \ldots
  \]
  so \( \hat{W}_{j,t} = \bar{W}_{j,t} \) if mod not needed: \( L_j - 1 \leq t < N \)
- if \( N - L_j \geq 0 \), unbiased estimator of \( \nu^2_{X}(\tau_j) \) is
  \[
  \hat{\nu}^2_{X}(\tau_j) \equiv \frac{1}{N - L_j + 1} \sum_{t=L_j-1}^{N-1} \bar{W}^2_{j,t} = \frac{1}{M_j} \sum_{t=L_j-1}^{N-1} \bar{W}^2_{j,t},
  \]
  where \( M_j \equiv N - L_j + 1 \)
- can also construct biased estimator of \( \nu^2_{X}(\tau_j) \):
  \[
  \hat{\nu}^2_{X}(\tau_j) \equiv \frac{1}{N} \sum_{t=0}^{N-1} \tilde{W}^2_{j,t} = \frac{1}{N} \left( \sum_{t=0}^{L_j-2} \tilde{W}^2_{j,t} + \sum_{t=L_j-1}^{N-1} \tilde{W}^2_{j,t} \right)
  \]
  1st sum in parentheses influenced by circularity
Estimation of Wavelet Variance: II

• biased estimator unbiased if \( \{X_t\} \) white noise

• biased estimator offers exact analysis of \( \hat{\sigma}^2 \); unbiased estimator need not

• biased estimator can have better mean square error (Greenhall et al., 1999; need to ‘reflect’ \( X_t \))
Statistical Properties of $\hat{\nu}_X^2(\tau_j)$

- Suppose $\{W_{j,t}\}$ Gaussian, mean 0 & spectrum $S_j(f)$
- Suppose square integrability condition holds:
  \[ A_j \equiv \int_{-1/2}^{1/2} S_j^2(f) df < \infty \text{ & } S_j(f) > 0 \]
  (holds for power law processes if $L$ large enough)
- Can show $\hat{\nu}_X^2(\tau_j)$ asymptotically normal with
  mean $\nu_X^2(\tau_j)$ & large sample variance $2A_j/M_j$
- Can estimate $A_j$ and use with $\hat{\nu}_X^2(\tau_j)$
  to construct confidence interval for $\nu_X^2(\tau_j)$
- Example
  - Fig. 13: clock errors $X_t \equiv X_t^{(0)}$ along with
    differences $X_t^{(i)} \equiv X_t^{(i-1)} - X_{t-1}^{(i-1)}$ for $i = 1, 2$
  - Fig. 14: $\hat{\nu}_X^2(\tau_j)$ for clock errors
  - Fig. 15: $\hat{\nu}_Y^2(\tau_j)$ for $\bar{Y}_t \propto X_t^{(1)}$
  - Haar $\hat{\nu}_Y^2(\tau_j)$ related to Allan variance $\sigma_Y^2(2, \tau_j)$:
    \[ \nu_Y^2(\tau_j) = \frac{1}{2} \sigma_Y^2(2, \tau_j) \]
Decorrelation of FD Processes

- $X_t$ ‘fractionally differenced’ if its spectrum is

$$S_X(f) = \frac{\sigma^2}{|2\sin(\pi f)|^{2\delta}},$$

where $\sigma^2 > 0$ and $-\frac{1}{2} < \delta < \frac{1}{2}$

- note: for small $f$, have $S_X(f) \approx C/|f|^{2\delta}$; i.e., power law with $\alpha = -2\delta$

- if $\delta = 0$, FD process is white noise

- if $0 < \delta < \frac{1}{2}$, FD stationary with ‘long memory’

- can extend definition to $\delta \geq \frac{1}{2}$
  - nonstationary $1/f$ type process
  - also called ARFIMA$(0,\delta,0)$ process

- Fig. 16: DWT of simulated FD process, $\delta = 0.4$
  (sample autocorrelation sequences (ACSs) on right)
DWT as Whitening Transform

• sample ACSs suggest $W_j \approx$ uncorrelated

• since FD process is stationary, so are $W_j$
  (ignoring terms influenced by circularity)

• Fig. 17: spectra for $W_j$, $j = 1, 2, 3, 4$

• $W_j$ & $W_{j'}$, $j \neq j'$, approximately uncorrelated
  (approximation improves as $L$ increases)

• DWT thus acts as a whitening transform

• lots of uses for whitening property, including:
  1. testing for variance changes
  2. bootstrapping time series statistics
  3. estimating $\delta$ for stationary/nonstationary
     fractional difference processes with trend
Estimation for FD Processes: I

- extension of work by Wornell; McCoy & Walden
- problem: estimate $\delta$ from time series $U_t$ such that
  \[ U_t = T_t + X_t \]
  where
  - $T_t \equiv \sum_{j=0}^r a_j t^j$ is polynomial trend
  - $X_t$ is FD process, but can have $\delta \geq \frac{1}{2}$
- DWT wavelet filter of width $L$ has embedded differencing operation of order $L/2$
- if $\frac{L}{2} \geq r + 1$, reduces polynomial trend to 0
- can partition DWT coefficients as
  \[ W = W_s + W_b + W_w \]
  where
  - $W_s$ has scaling coefficients and 0s elsewhere
  - $W_s$ has boundary-dependent wavelet coefficients
  - $W_w$ has boundary-independent wavelet coefficients
Estimation for FD Processes: II

- since \( U = W^T W \), can write
  \[
  U = W^T (W_s + W_b) + W^T W_w \equiv \tilde{T} + \tilde{X}
  \]
- Fig. 18: example with fractional frequency deviates
- can use values in \( W_w \) to form likelihood:
  \[
  L(\delta, \sigma^2_\epsilon) \equiv \prod_{j=1}^{J_0} \prod_{t=1}^{N_j'} \frac{1}{(2\pi \sigma^2_j)^{1/2}} e^{-W_{j,t+L_j'-1}/(2\sigma^2_j)}
  \]
  where
  \[
  \sigma^2_j \equiv \int_{-1/2}^{1/2} \mathcal{H}(f) \frac{\sigma^2_\epsilon}{|2\sin(\pi f)|^{2\delta}} df;
  \]
  and \( \mathcal{H}(f) \) is squared gain for \( h_{j,l} \)
- leads to maximum likelihood estimator \( \hat{\delta} \) for \( \delta \)
- works well in Monte Carlo simulations
- get \( \hat{\delta} \equiv 0.39 \pm 0.03 \) for fractional frequency deviates
DWT-based Signal Extraction: I

• DWT analysis of $X$ yields $W = \mathcal{W}X$
• DWT synthesis $X = \mathcal{W}^T W$ yields
  - multiresolution analysis (MRA)
  - estimator of ‘signal’ $D$ hidden in $X$:
    * modify $W$ to get $W'$
    * use $W'$ to form signal estimate:
      $$\hat{D} \equiv \mathcal{W}^T W'$$
• key ideas behind wavelet-based signal estimation
  - DWT can isolate signals in small number of $W_n$’s
  - can ‘threshold’ or ‘shrink’ $W_n$’s
• key ideas lead to ‘waveshrink’
  (Donoho and Johnstone, 1995)
DWT-based Signal Extraction: II

- thresholding schemes involve
  1. computing \( \mathbf{W} \equiv \mathbf{WX} \)
  2. defining \( \mathbf{W}^{(t)} \) as vector with \( n \)th element
     \[
     \mathbf{W}^{(t)}_n = \begin{cases}
     0, & \text{if } |W_n| \leq \delta; \\
     \text{some nonzero value}, & \text{otherwise},
     \end{cases}
     \]
     where nonzero values are yet to be defined
  3. estimating \( \mathbf{D} \) via \( \hat{\mathbf{D}}^{(t)} \equiv \mathbf{W}^T \mathbf{W}^{(t)} \)

- simplest scheme is 'hard thresholding:'
  \[
  \mathbf{W}^{(ht)}_n = \begin{cases}
  0, & \text{if } |W_n| \leq \delta; \\
  W_n, & \text{otherwise}.
  \end{cases}
  \]

Fig. 19: solid line (‘kill/keep’ strategy)

- alternative scheme is ‘soft thresholding:'
  \[
  \mathbf{W}^{(st)}_n = \text{sign} \{W_n\} \ (|W_n| - \delta)_+,
  \]
  where
  \[
  \text{sign} \{W_n\} \equiv \begin{cases}
  +1, & \text{if } W_n > 0; \\
  0, & \text{if } W_n = 0; \\
  -1, & \text{if } W_n < 0.
  \end{cases}
  \]
  \[
  (x)_+ \equiv \begin{cases}
  x, & \text{if } x \geq 0; \\
  0, & \text{if } x < 0.
  \end{cases}
  \]

Fig. 19: dashed line
DWT-based Signal Extraction: III

- third scheme is ‘mid thresholding:

\[ W_n^{(mt)} = \text{sign}\{W_n\} (|W_n| - \delta)_{++}, \]

where

\[ (|W_n| - \delta)_{++} \equiv \begin{cases} 2(|W_n| - \delta)_+, & \text{if } |W_n| < 2\delta; \\ |W_n|, & \text{otherwise} \end{cases} \]

Fig. 19: dotted line

- Q: how should \( \delta \) be set?

- A: universal’ threshold (Donoho & Johnstone, 1995) (lots of other answers have been proposed)

  - specialize to model \( X = D + \epsilon, \)
  where \( \epsilon \) is Gaussian white noise with variance \( \sigma^2 \)
  - ‘universal’ threshold: \( \delta_U \equiv \sqrt{2\sigma^2 \log(N)} \)
  - rationale for \( \delta_U \):
    * suppose \( D = 0 \) & hence \( W \) is white noise also
    * as \( N \to \infty \), have
    \[
    P \left[ \max_n |W_n| \leq \delta_U \right] \to 1
    \]
    so all \( W^{(ht)} = 0 \) with high probability
    * will estimate correct \( D \) with high probability
DWT-based Signal Extraction: IV

• can estimate $\sigma^2$ using median absolute deviation (MAD):

$$\hat{\sigma}_{\text{MAD}} \equiv \frac{\text{median}\left\{|W_{1,0}|, |W_{1,1}|, \ldots, |W_{1,\frac{N-1}{2}}|\right\}}{0.6745},$$

where $W_{1,t}$’s are elements of $W_1$

• Fig. 20: application to NMR series

• has potential application in dejamming GPS signals (with roles of ‘signal’ and ‘noise’ swapped!)
Web Material and Books

• Wavelet Digest
  
  http://www.wavelet.org/

• MathSoft’s wavelet resource page
  
  http://www.mathsoft.com/wavelets.html

• books
  
  
  
  
  
  http://www.staff.washington.edu/dbp/wmtsa.html
  
Software

- **Matlab**
  - **Wavelab** (free):
    
    http://www-stat.stanford.edu/~wavelab
  
  - **WaveBox** (commercial):
    
    http://www.toolsmiths.com/

- **Mathcad** Wavelets Extension Pack (commercial):
  
  http://www.mathsoft.com/mathcad/ebooks/wavelets.asp

- **S-Plus** software
  
  - **Wavethresh** (free):
    
    http://lib.stat.cmu.edu/S/wavethresh
  
  - **S+Wavelets** (commercial):
    
    http://www.mathsoft.com/splsprod/wavelets.html
Time series analysis is therefore an important topic for many different applications. For series that satisfy certain properties, such as stationarity and regular spacing of observations, there is much literature on well-established analysis methods. This thesis is primarily concerned with analysis methods for nonstationary time series with characteristics that deviate from the standard assumptions. We consider the analysis of bivariate signals, and time series that are observed with irregular sampling intervals. Chapter 2 reviews the literature in the area of time series analysis using wavelets. It summarises the basic concepts from wavelet theory, including.